AGH	**AGH University of Science and Technology in Cracow** **Department of Electronics**

Laboratory Manual
Physics_1

Title:

Uncertainties of measurements

1. Standard deviation

To find the uncertainty in measurements, we often calculate the **standard deviation**, or σ, of the measured value. Standard deviation is a measure of the variation of *N* data points ($x_1 ... x_n$) about an average value, $\bar{x}$, and is typically called the uncertainty of a measured result.

The **average or mean value**, $\bar{x}$, of a set of *n* measurements is equal to:

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{1}{n}\Sigma x_i$$

Once the mean value of the measurements is determined, it is helpful to define how much the individual measurements are scattered around about the mean. The **deviation**, $\bar{d}$, of any measurement, x_i, from the mean $\bar{x}$ is given by :

$$d_i = x_i - \bar{x}$$

Since the deviation may be either positive or negative, it is often more useful to use the **mean deviation**, or $\bar{d}$, to determine the uncertainty of the measurement. This is found by averaging the absolute deviations, $|d_i| = |x_i - \bar{x}|$; that is,

$$\bar{d} = \frac{1}{n}\Sigma|d_i|$$

To avoid the use of absolute values we can use the square of the deviation, d_i^2, to more accurately determine the uncertainty of our measurement. The standard deviation, σ, (sometimes called the **root-mean square**) is given by

$$\sigma = \sqrt{\frac{1}{n}\Sigma d_i^2}$$

It can be shown that for a small number of measurements, this equation becomes:

$$\sigma = \sqrt{\frac{1}{n-1}\Sigma d_i^2}$$

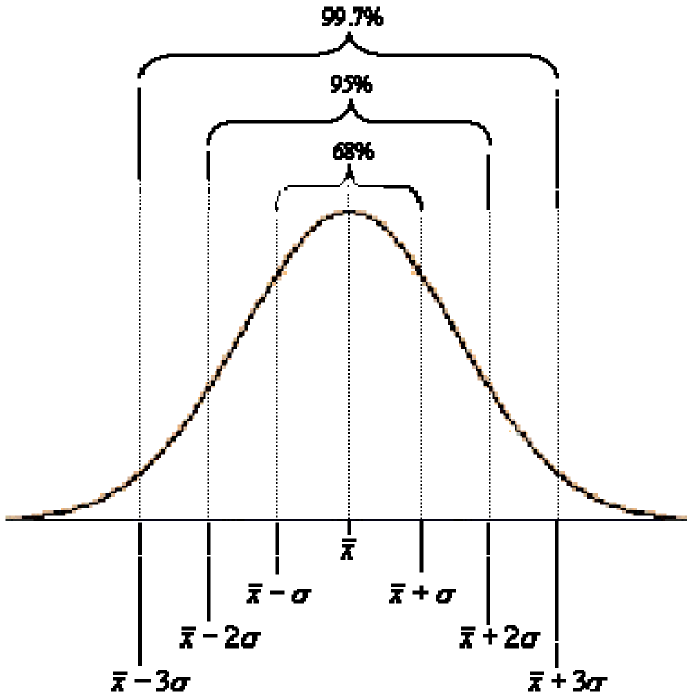
where n is replaced by $n-1$.

The experimental result, E_x, can then be written as

$$E_x = \bar{x} \pm \sigma$$

where, σ gives the measure of the precision of the measurement.

Notice the standard deviation is always positive and has the same units as the mean value. It can be shown that there is a 68% likelihood that an individual measurement will fall within one standard deviation $(\pm\sigma)$ of the true value. Furthermore, it can be shown that there also exists a 95% likelihood that an individual measurement will fall within two standard deviations $(\pm 2\sigma)$ of the true value, and a 99.7% likelihood that it will fall within $(\pm 3\sigma)$ of the true value.



Some useful expressions:

σ expresses the standard deviation for the single measurement x_i, so often the notation σ_x is used.
It can be proved, that standard deviation of the mean $\bar{x}$ is equal to::

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

When dealing with uncertainties based on a large collection of numbers the manipulation of measured quantities and the error associated with each quantity will contribute to the error in the final answer. The following formulae are useful:

$\bar{z} = \bar{x} + \bar{y} \qquad \sigma_z = \sqrt{(\sigma_x)^2 + (\sigma_y)^2}$

$\bar{z} = \bar{x} - \bar{y} \qquad \sigma_z = \sqrt{(\sigma_x)^2 + (\sigma_y)^2}$

2. Linear regression

If the relationship between two sets of data (*x* and *y*) is linear and the data is plotted (*y* versus *x*) the result is a straight line. This relationship is known as having a **linear correlation** and follows the equation of a straight line $y = ax + b$.

We can apply a statistical treatment known as **linear regression** to the data (x_i, y_i) in order to determine the constants **a** and **b** which are called regression coefficients: ***a*** is the slope and ***b*** is the y-intercept for the line which the best fits the data.

Given a set of data (x_i, y_i) with *n* data points, the regression coefficients can be determined using the following:

$$a = \frac{n\sum x_i y_i - \sum x_i \cdot \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$b = \frac{\sum x_i^2 \cdot \sum y_i - \sum x_i \cdot \sum x_i \cdot y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

The standard deviations of the linear coefficients σ_a=Δa and σ_b==Δb are given by expressions:

$$\Delta a = \sqrt{\frac{n}{n-2}} \cdot \sqrt{\frac{\sum \left[y_i - (ax_i + b)\right]^2}{n\sum x_i^2 - \left(\sum x_i\right)^2}}$$

$$\Delta b = \Delta a \cdot \sqrt{\frac{\sum x_i^2}{n}}$$

It may appear that the above equations are quite complicated , however upon inspection, we can see that their components are nothing more than simple algebraic manipulations of the columns of data. The table below is useful for these calculations.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$	a	b	$y_i - (ax_i + b)$	$(y_i - (ax_i + b))^2$
x_1	y_1							
.	.	.						
.	.	.						
.	.	.						
.	.	.						
.	.	.						
.	.							
x_n	y_n							
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$				$\sum (y_i - (ax_i + b))^2$

When y-intercept is equal to zero that means **b=0** and $y = ax$ we should use one-dimensional linear regression which coefficients can be determined using the following equations:

$$a = \frac{\sum_{i=1}^{n} x_i \cdot y_i}{\sum_{i=1}^{n} x_i^{\,2}}$$

and the standard deviation of the linear coefficients $\sigma_a = \Delta a$ is given by the expression:

$$\Delta a = \sqrt{\frac{1}{n-1} \frac{\sum_{i=1}^{n} \left[(a \cdot x_i - y_i)^2 \right]}{\sum_{i=1}^{n} x_i^{\,2}}}$$

3. Total differential method

Suppose f is a function of several independent variables, i.e. $f(x_1, x_2, x_3, \ldots, x_k)$ and each of these variables has its own uncertainty $\Delta x_1, \Delta x_2, \ldots, \Delta x_k$.

Absolute error of such measurements Δf can be calculated using the method of total differential of a function f**.** Take the differential of f with respect to several variables:

$$\Delta f = \left|\frac{\partial f}{\partial x_1}\right| \cdot \Delta x_1 + ... + \left|\frac{\partial f}{\partial x_n}\right| \cdot \Delta x_n$$

where $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_k}$ are partial derivatives. A partial derivative means that the derivative is taken with respect to one variable, while all the other variable are considered constant.

4. Logarithmic derivative method

When the function f is the product of variables x_1 .... x_k, the log derivative method is used to find the uncertainty of f.

$$f = x_1^{a1} \cdot x_2^{a2} \cdot ..... \cdot x_k^{a_k}$$

Take the natural log of each side:.

$$\ln f = a_1 \ln x_1 + a_2 \ln x_2 + ... + a_k \ln x_k$$

Then differentiate:

$$\frac{\Delta f}{f} = a_1 \frac{\Delta x_1}{x_1} + a_2 \frac{\Delta x_2}{x_2} + ... + a_k \frac{\Delta x_k}{x_k}$$

Let's take the absolute values of a_1 , ..., a_k . because uncertainties always sum up and never subtract.

Relative uncertainty::

$$\sigma = \frac{\Delta f}{f} = |a_1| \frac{\Delta x_1}{x_1} + |a_2| \frac{\Delta x_2}{x_2} + ... + |a_k| \frac{\Delta x_k}{x_k}$$

Example 1

Calculate the uncertainty of mobility of the carriers μ using the method of logarithmic derivative:

$$\mu = \frac{al}{dB_z}$$

Calculate logarithm of the both sides of the equation:

$$\ln \mu = \ln a + \ln l - \ln B_z - \ln d$$

Derivate both sides of the equation:

$$\left|\frac{\Delta \mu}{\mu}\right| = \left|\frac{\Delta a}{a}\right| + \left|\frac{\Delta l}{l}\right| + \left|\frac{\Delta B_z}{B_z}\right| + \left|\frac{\Delta b}{b}\right|$$

where:

$\Delta a = \sigma_a$ is the uncertainty of the regression coefficient a

$\Delta l = 0{,}2 \cdot 10^{-3} m$, $\Delta b = 0{,}1 \cdot 10^{-3} m$ are the geometrical uncertainties of the probe,

$\Delta B_z =$ is the uncertainty of reading B_z from the electromagnet characteristic.

Example 2

Calculate the uncertainty of concentration of the carriers *n* using the total differential method

$$n = \frac{1}{R_H e}$$

Remember: e= 1.6 x 10^{-19} C.

$$\Delta n = \left| \frac{\partial n}{\partial R_H} \right| \cdot \Delta R_H$$

$$\frac{\partial n}{\partial R_H} = -\frac{1}{e \cdot R_H^2}$$

$$\Delta n = \left| \frac{1}{e \cdot R_H^2} \right| \cdot \Delta R_H$$

5. Propagation of uncertainties principle

The uncertainty of the function $y = f(x_1, x_2, \ldots, x_n)$ can be calculate from the propagation of the uncertainty principle, as the geometrical sum of the partial derivatives:

$$\Delta y = \sqrt{\left[\frac{\partial y}{\partial x_1}\Delta x_1\right]^2 + \left[\frac{\partial y}{\partial x_2}\Delta x_2\right]^2 + \ldots + \left[\frac{\partial y}{\partial x_n}\Delta x_n\right]^2}$$

Example 1

Refractive index measurement from the equation is equal to:

$$n = \frac{D}{d_1 - d_2},$$

From the measurement we have values of: D, ΔD, d_1, Δd_1, d_2 and Δd_2. Thus using propagation of uncertainties principle we can calculate Δn as follows:

$$\Delta n = \sqrt{\left[\frac{\partial n}{\partial D}\Delta D\right]^2 + \left[\frac{\partial n}{\partial d_1}\Delta d_1\right]^2 + \left[\frac{\partial n}{\partial d_2}\Delta d_2\right]^2}$$

Where

$\frac{\partial n}{\partial D} = \frac{1}{d_1 - d_2}$ and $\frac{\partial n}{\partial d_1} = \frac{D}{(d_1 - d_2)^2}$ and $\frac{\partial n}{\partial d_2} = \frac{D}{(d_1 - d_2)^2}$ thus:

$$\Delta n = \sqrt{\left[\frac{1}{d_1 - d_2}\Delta D\right]^2 + \left[\frac{D}{(d_1 - d_2)^2}(\Delta d_1 + \Delta d_2)\right]^2}$$

If, $D = 1.58$ mm, $d_1 = 1.26$ mm, $d_2 = 0.18$ mm then $n = 1.58/1.08 =$ **1.46**296293 and

$\Delta D = 0.01$, $\Delta d_1 = 0.01$ and $\Delta d_2 = 0.02$ then

$$\Delta n = \sqrt{\left(\frac{0.01}{1.08}\right)^2 + \left(\frac{1.58}{1.08}\right)^2 \left(0.01^2 + 0.02^2\right)} = 0.03$$

Updated : Barbara Dziurdzia 14.02.2009 r., Barbara Dziurdzia 27.03.2014 r., Witold Skowroński 9.03.2015