# Uncertainty in measurements

# Uncertainty in Measurements

In October 1992, a new policy on expressing measurement uncertainty was instituted at NIST, National Institute of Standards and Technology.

Elaboration of ***Guide to Expression of Uncertainty in Measurement*** by International Organization for Standardization, ISO, 1993

Applicable to results associated with:

- international comparisons of measurement standards,
- basic research,
- applied research and engineering,
- calibrating client measurement standards,
- certifying standard reference materials, and
- generating standard reference data.

http://physics.nist./gov/Uncertainty

Wyrażanie Niepewności Pomiaru. Przewodnik. Warszawa, Główny Urząd Miar 1999

# MEASUREMENT

The result of a measurement is only an approximation or estimate of the value of the specific quantity subject to measurement, the measurand which can be classified as:

- simple, or
- complex

Example: Mathematical pendulum, l – the length, T – period are simple measurands; measured directly

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Determination of gravitational acceleration : g-complex measurand

# MEASUREMENT

**In the course of measurements values different from those predicted by theory are obtained. The source of discrepancies between theory and experiment can be traced back to imperfections due to:**

**-experimentalist,**

**-measuring equipment,**

**-object measured**

**More perfect the experiment is made, discrepancies decrease. Error, uncertainty can be reduced.**

## Result of a measurement should be given in one of the following forms:

$$F = (98 \pm 3) \cdot 10^3 \, C$$

$$g = 9{,}866(28) \, m/s^2$$

Example: In an experiment, the electrochemical equivalent k was found to be:

**How one can express this result?**

k=0,00109 | 63 g/C

Δk=0,0003 | 47 g/C

significant digits | Non-significant digits

**Answer. k= (0,00110 ± 0,00004) g/C or k= 0,00110(4) g/C**

# Uncertainty / error

## Absolute error

$$\Delta x_i = x_i - x_0 \quad (1)$$

**$x_i$ – experimental result, $x_0$ – real value**

## Relative error:

$$\delta = \frac{\Delta x_i}{x_0} \quad (2)$$

**Note: real values of measurand are unknown in most cases**

# Uncertainty

Quantities given by formulas (1) and (2) are singular realization of random variable which is why they cannot be treated by theory of uncertainty. Practically, we do not know real values and estimate **uncertainties,** due to dispersion of results, from the laws of **statistics**.

**Uncertainty** is

- a **parameter** related to the result of measurements,
- characterized by **dispersion**
- assigned to the measurand in a **justified way**.

# Absolute and relative uncertainty

**Absolute uncertainty u** is expressed in the same units as a measurand

Symbols: u or u(x) or u(concentration of NaCl)

**Relative uncertainty $u_r(x)$** the ratio of absolute uncertainty to the measured value:

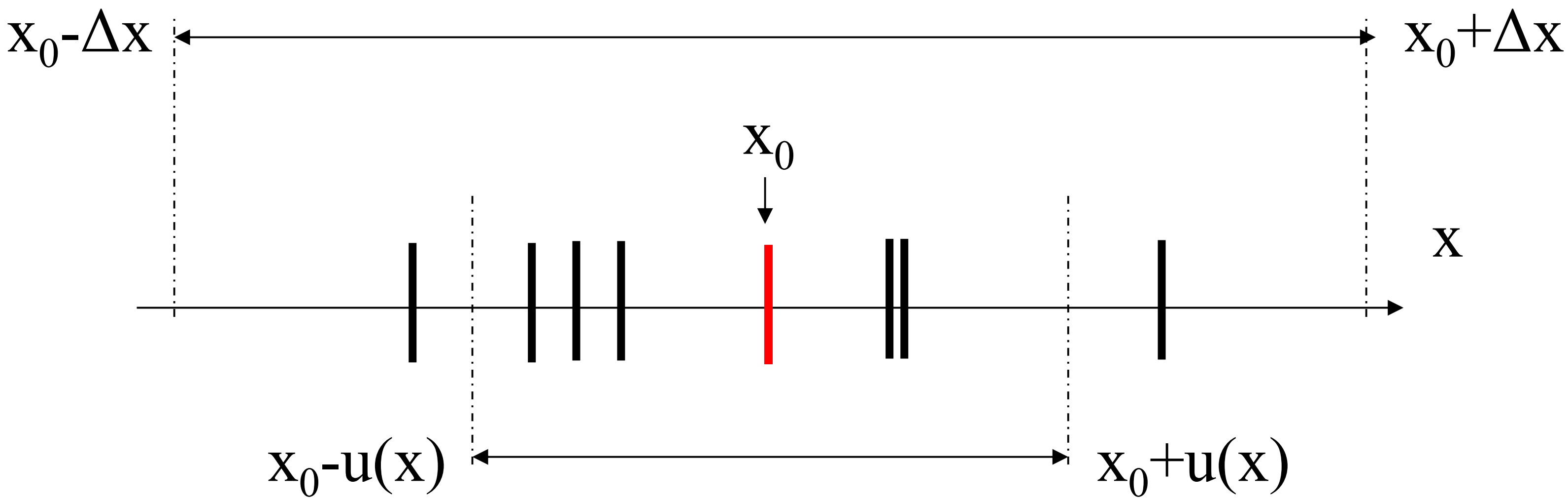
$$u_r(x) = \frac{u(x)}{x}$$

Relative uncertainty has no units and can be expressed in %

# Measures of uncertainty

There exist two measures:

- ❑ standard uncertainty u(x)
- ❑ maximum uncertainty Δx



# Standard uncertainty

Generally accepted and suggested.

1. Distribution of random variable $x_i$, with a dispersion around the average $\overline{x}$ is characterized by **standard deviation** defined as:

$$\sigma = \lim_{n \to \infty} \sqrt{\frac{\sum (x_i - \overline{x})^2}{n}}$$

2. Exact values of standard deviation are unknown. **Standard uncertainty** represents an estimate of **standard deviation.**

# Maximum uncertainty

Within this interval:

$$x_0 - \Delta x < x_i < x_0 + \Delta x$$

all the results $x_i$, will fall.

**Deterministic measure.**

It is recommended to replace the maximum uncertainty by a standard uncertainty:

$$u(x) = \frac{\Delta x}{\sqrt{3}}$$

# Classification of errors aa1 aa2

Results of measurements follow some regular patterns i.e. they are distributed in a way typical for random variables. According to distribution functions and sources of errors one can distinguish:

**Gross errors** (mistakes) that have to be eliminated

**Systematic error** that can be reduced when improving the measurement

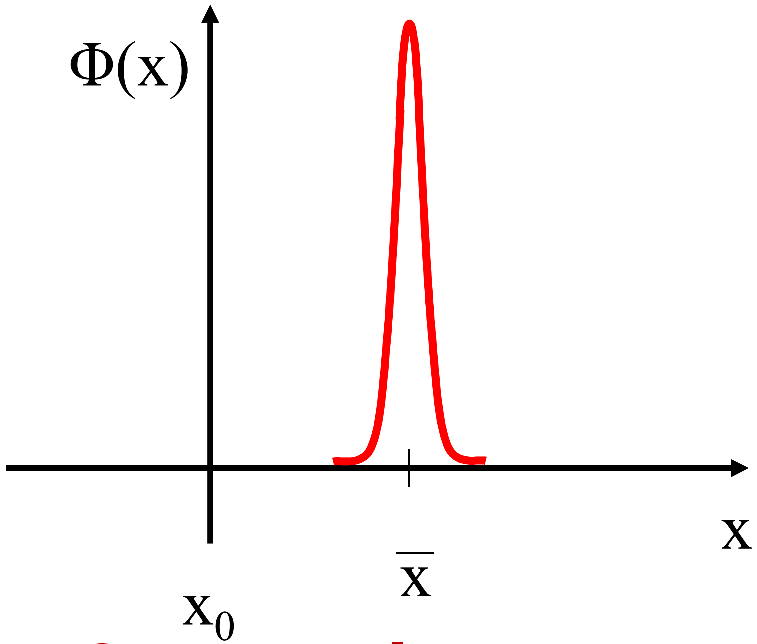
**Random errors** that result from numerous random contributions and cannot be eliminated; they should be treated within the formalism of statistics and probabilistics.

## Slajd 12

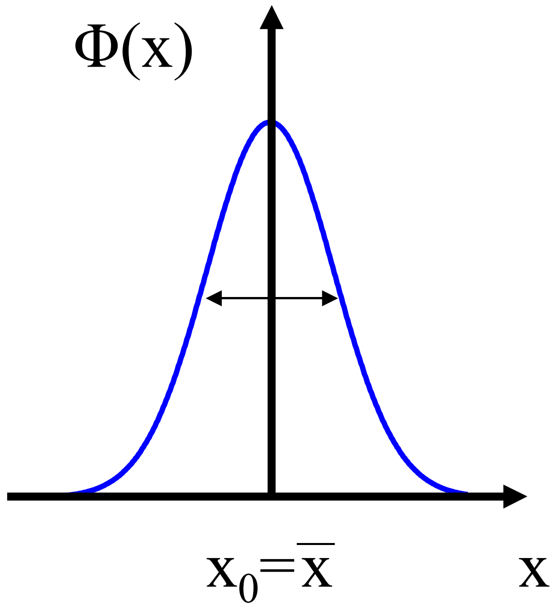
**aa1** aaa; 31.03.2015

**aa2** aaa; 31.03.2015

# Distribution functions




**Systematic error**




**Random error – Gauss distribution function**

Φ(x) – probability density function

# Analysis of uncertainties

**Type A**

All methods that use statistical approach:

- large number of repetitions is required
- applies to random sources of errors

**Type B**

Is based on scientific estimate performed by the experimentalist that has to use all information on the measurement and the source of its uncertainty

- applies when the laws of statistics cannot be used
- for a systematic error or for a single result of measurement

# TYPE A

AGH

Example:

We have performed a series of measurements getting the following results $x_1, x_2, \ldots x_n$. In such a sample that can be considered as big some of the results are the same; $n_k$ is a number of random experiments, in which the same result $x_k$ has occurred.

$n_k/n$ is a frequency of the result

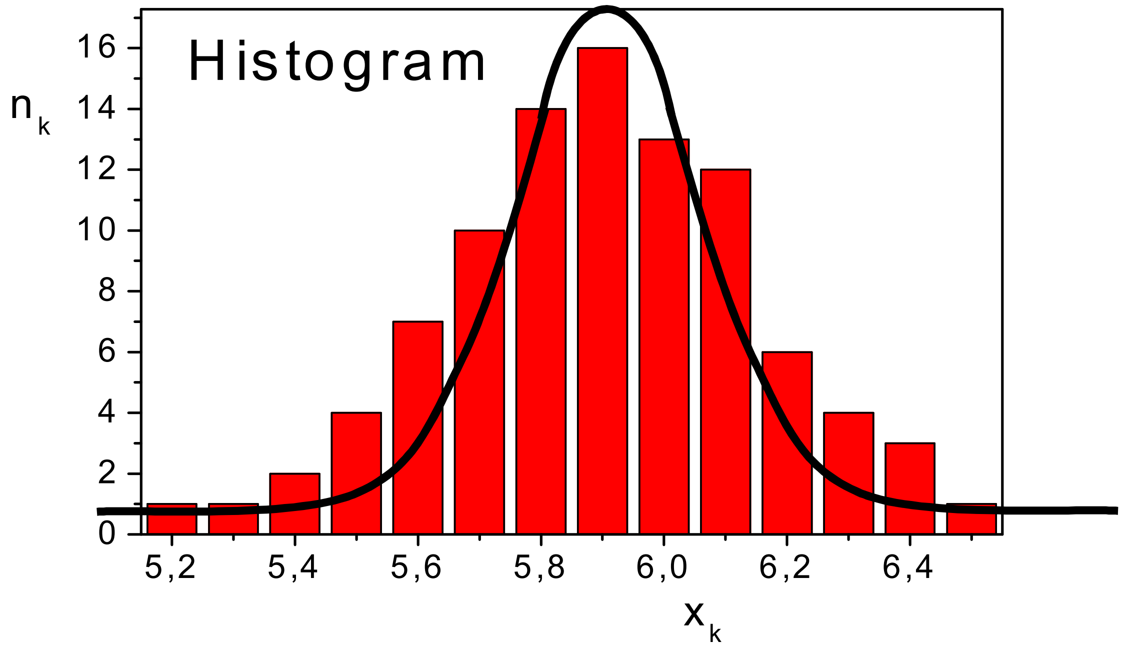
| $x_k$ | $n_k$ | $n_k/n$ |
|---|---|---|
| 5,2 | 1 | 0,011 |
| 5,3 | 1 | 0,011 |
| 5,4 | 2 | 0,021 |
| 5,5 | 4 | 0,043 |
| 5,6 | 7 | 0,075 |
| 5,7 | 10 | 0,106 |
| 5,8 | 14 | 0,149 |
| 5,9 | 16 | 0,170 |
| 6,0 | 13 | 0,138 |
| 6,1 | 12 | 0,128 |
| 6,2 | 6 | 0,064 |
| 6,3 | 4 | 0,043 |
| 6,4 | 3 | 0,032 |
| 6,5 | 1 | 0,011 |
| **Sum** | 94 | |

# Analysis of data

Arithmetic average

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\bar{x}$=5,9




Standard uncertainty

$$\sigma = u(x) = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

σ=0,2

Standard uncertainty of the average

$$u(\bar{x}) = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n(n-1)}}$$

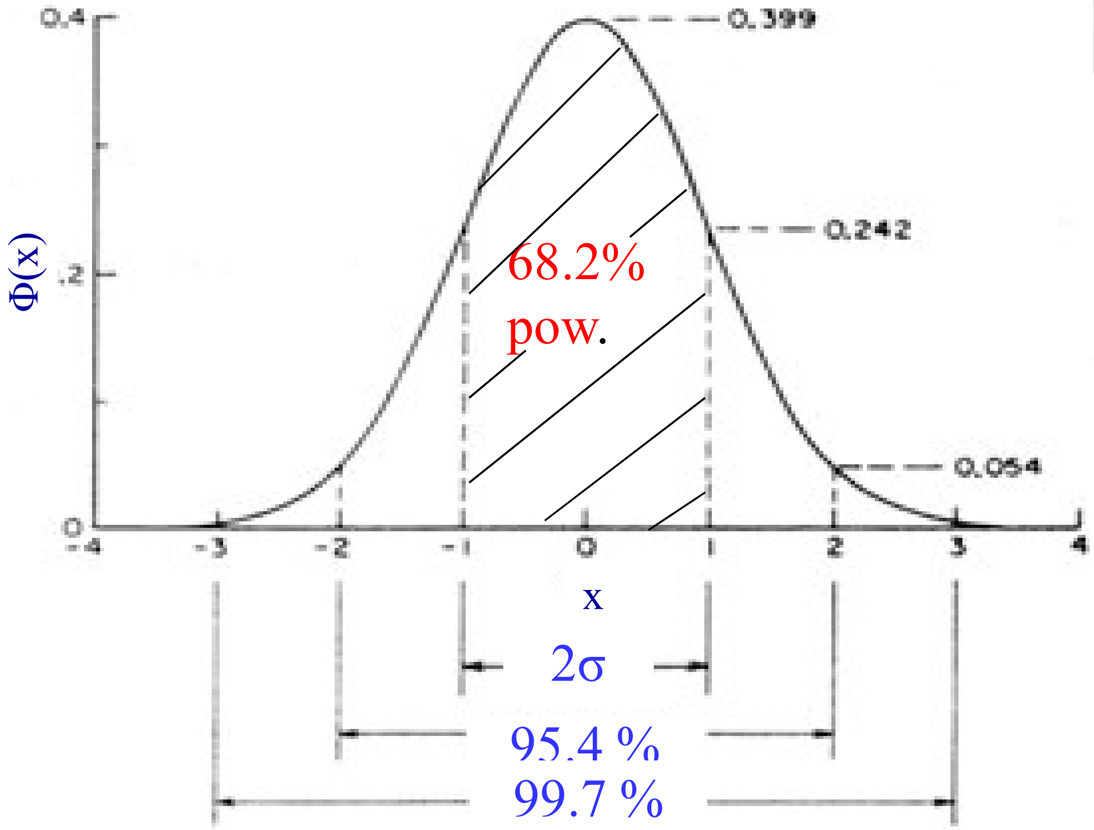
# Gauss distribution function

***Probability density function for the result x or its error $\Delta$x according to Gauss***

$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

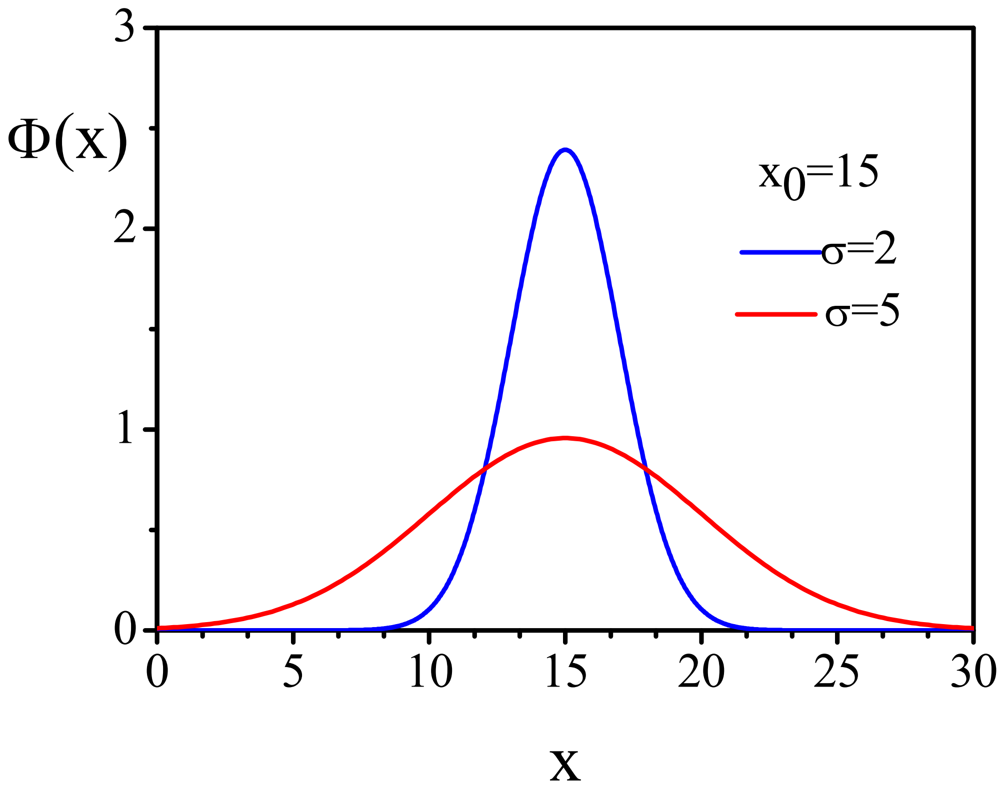
**$x_0$ is the most probable result and can be represented by the arithmetic average, $\sigma$ is standard deviation, $\sigma^2$ is variance**

# Normal distribution




Within the interval $x_0$-σ < $x$ < $x_0$+σ we find 68.2 % (2/3),
For $x_0$-2σ < $x$ < $x_0$+2σ - 95.4 %
For $x_0$-3σ < $x$ < $x_0$+3σ - 99.7 %
of all results

# Gauss distribution function




Bigger σ means higher scatter of the results around its average, smaller precision.

# TYPE B

# TYPE B

A type B evaluation of standard uncertainty is usually based on scientific judgement using all the relevant information available, which may include:

- previous measurement data,
- experience with, or general knowledge of, the behavior and property of relevant materials and instruments,
- manufacturer's specification
- data provided in calibration and other reports
- uncertainties assigned to reference data taken from handbooks

**Type A evaluations of uncertainty based on limited data are not necessarily more reliable than soundly based Type B evaluations.**

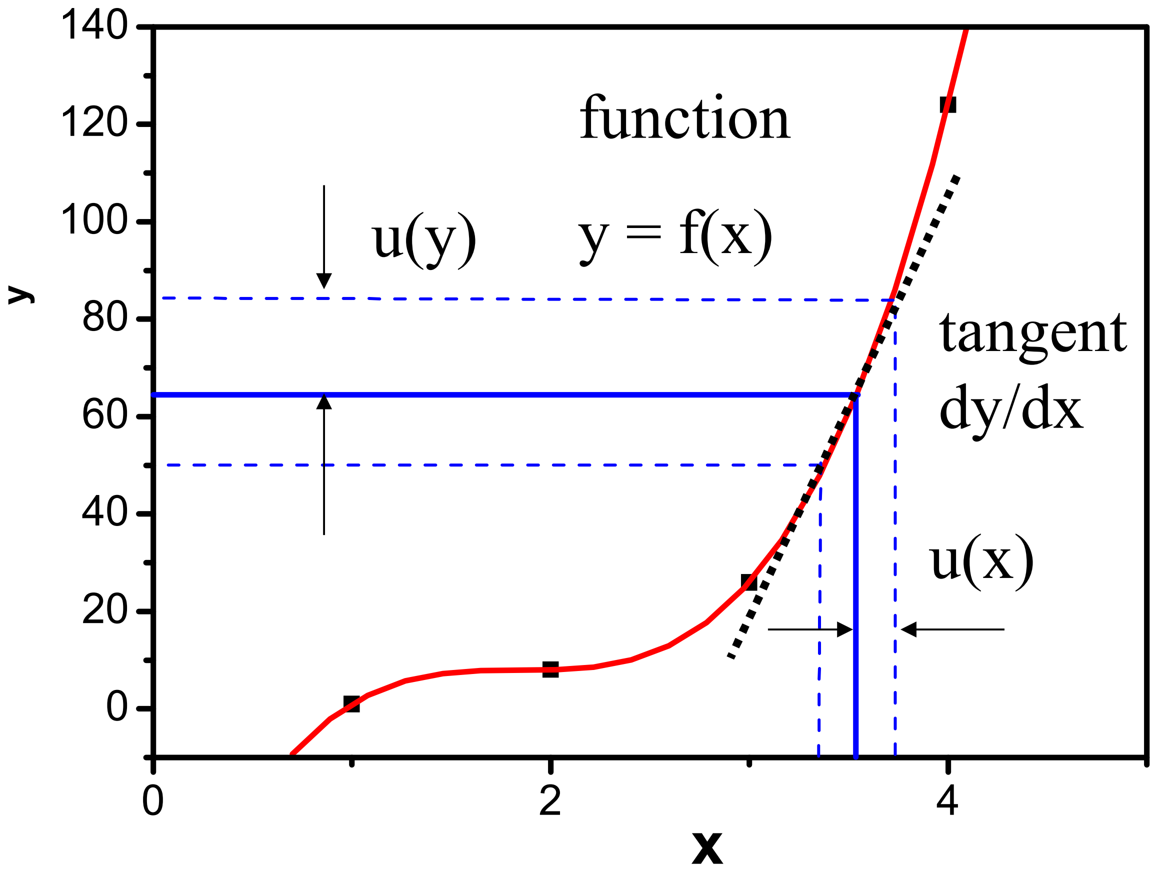
# TYPE B

Most often the type **B** deals with evaluation of uncertainty resulting from a finite accuracy of an instrument.

**Example: Type B uncertainty of pendulum length measurement.**

Using a ruler the following results were obtained: L=140 mm, u(L)=1 mm (elemental scale interval), $u_r(L)=u(L)/L=1/140$, percentage uncertainty 0,7%

# Uncertainty of complex measurand – propagation of errors




$$u(y) = \frac{dy}{dx} u(x)$$

# Total differential

For a complex measurand $y=f(x_1,x_2,...x_n)$ under the assumption that $\Delta x_1$ , $\Delta x_2$ , ... $\Delta x_n$ are small as compared with measured $x_1,x_2$, ... $x_n$, maximum uncertainty of y can be calculated from the differential calculus :

$$\Delta y = \left|\frac{\partial y}{\partial x_1}\right| \left|\Delta x_1\right| + \left|\frac{\partial y}{\partial x_2}\right| \left|\Delta x_2\right| + \dots + \left|\frac{\partial y}{\partial x_n}\right| \left|\Delta x_n\right| \quad (3)$$

# Law of propagation of uncertainties

Standard uncertainty of complex measurand $y=f(x_1,x_2,...x_n)$ can be calculated from the law of propagation of uncertainties as a geometric sum of partial differentials.

$$u_c(y)=\sqrt{\left[\frac{\partial y}{\partial x_1}u(x_1)\right]^2+\left[\frac{\partial y}{\partial x_2}u(x_2)\right]^2+...+\left[\frac{\partial y}{\partial x_n}u(x_n)\right]^2}$$

$$u_{cr}(y)=\frac{u_c(y)}{y}$$

# Total differential applied to systematic error

*We measure U and I, then R is determined from* $R = U / I$

*Maximum uncertainty of R (eq. 3)*

$$\Delta R = \left|\frac{\partial R}{\partial U}\right|\left|\Delta U\right| + \left|\frac{\partial R}{\partial I}\right|\left|\Delta I\right| \qquad \frac{\partial R}{\partial U} = \frac{1}{I} \qquad \frac{\partial R}{\partial I} = -\frac{U}{I^2}$$

Absolute uncertainty

$$\Delta R = \frac{1}{I}\Delta U + \frac{U}{I^2}\Delta I$$

Relatice uncertainty

$$\frac{\Delta R}{R} = \frac{\Delta U}{U} + \frac{\Delta I}{I}$$

Accuracy of instruments for ΔU and ΔI measurements affects the uncertainty in R

# Example

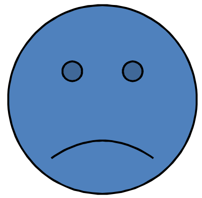
In a certain experiment one determines gravitational acceleration g on Earth by measuring the period T and length L of a mathematical pendulum. Directly measured length is reported as 1.1325±0.0014 m. Independently estimated relative uncertainty of period measurement is 0.06%, i.e.,

$$u_r(T) = \frac{u(T)}{T} = 6 \cdot 10^{-4}$$

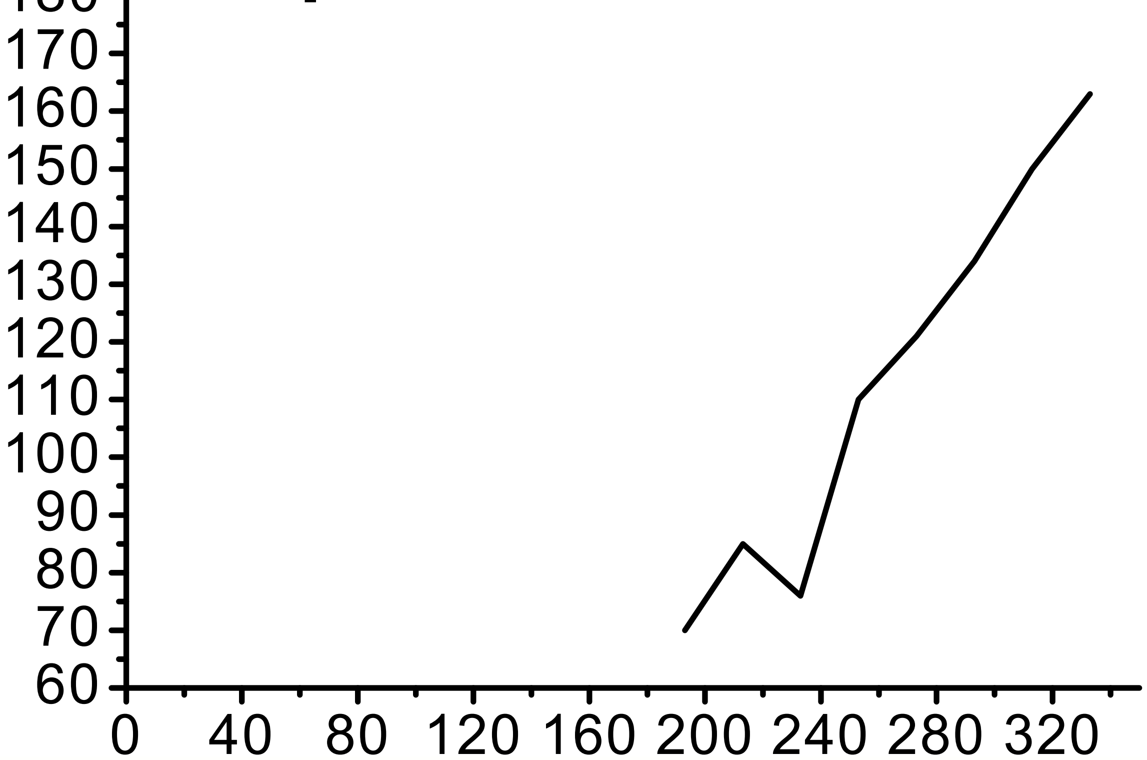
Calculate the relative uncertainty of g assuming that the uncertainties of L and T are independent and result from random sources of errors.

# Rules applied to data plotting

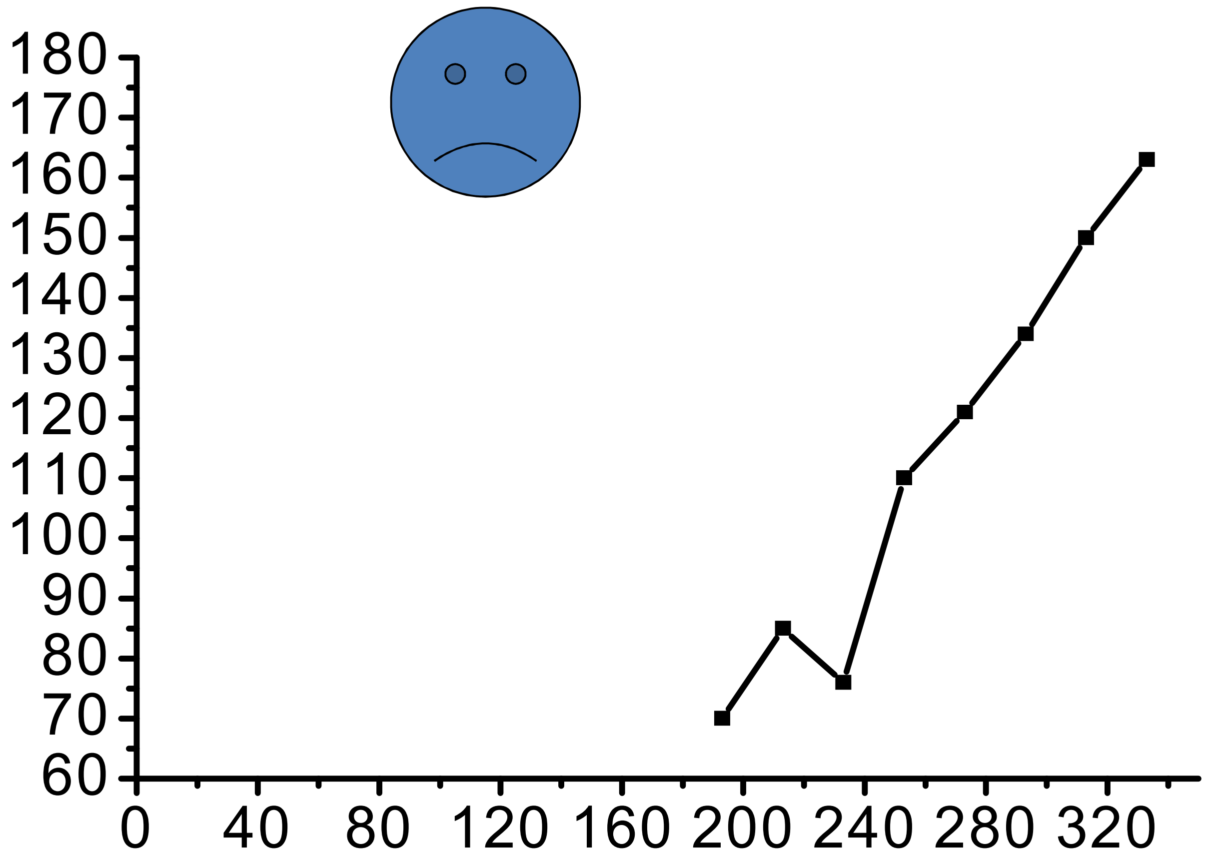
## Is this graph made according to the rules?



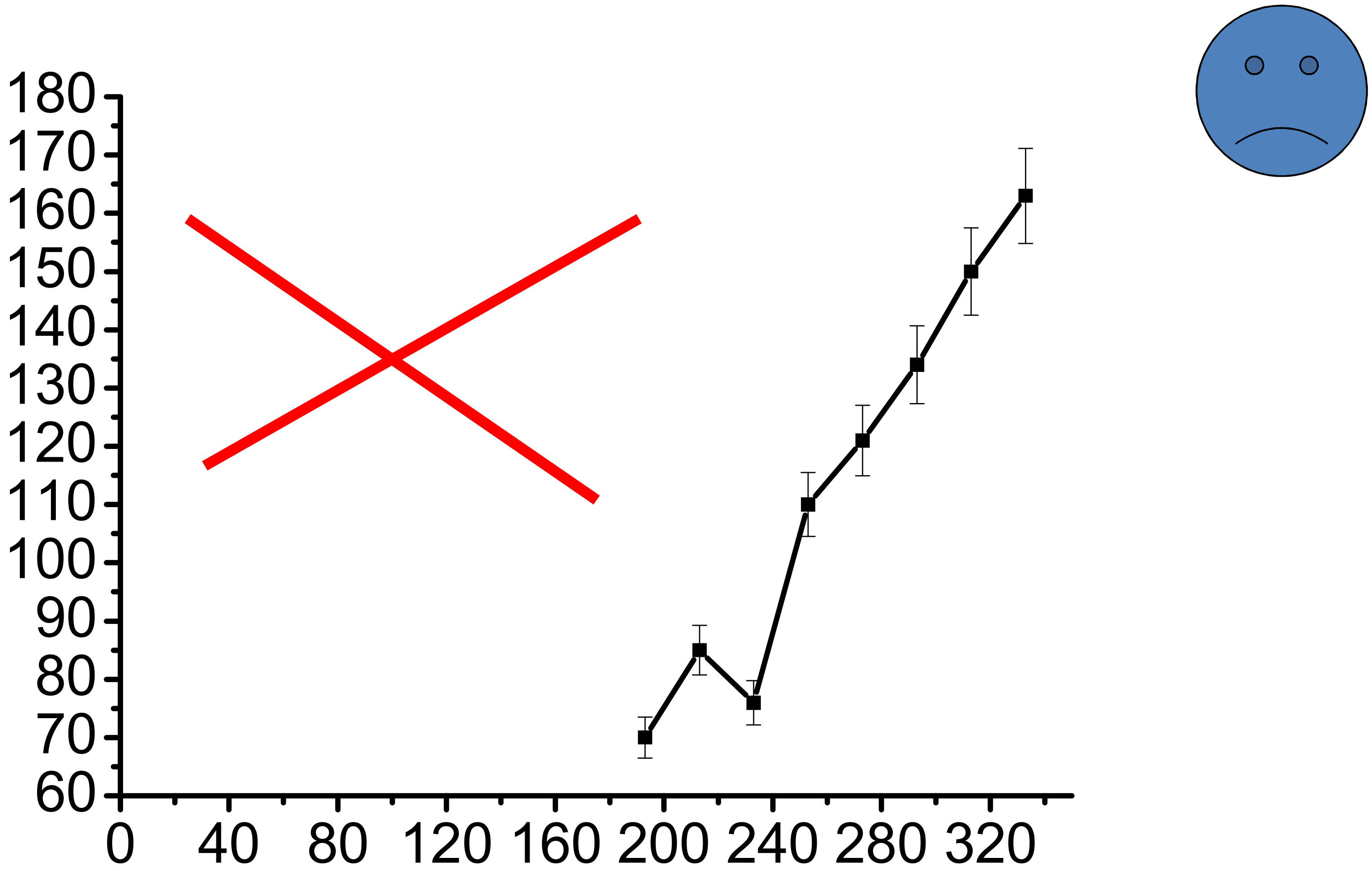
**1. Mark the experimental points!!!**



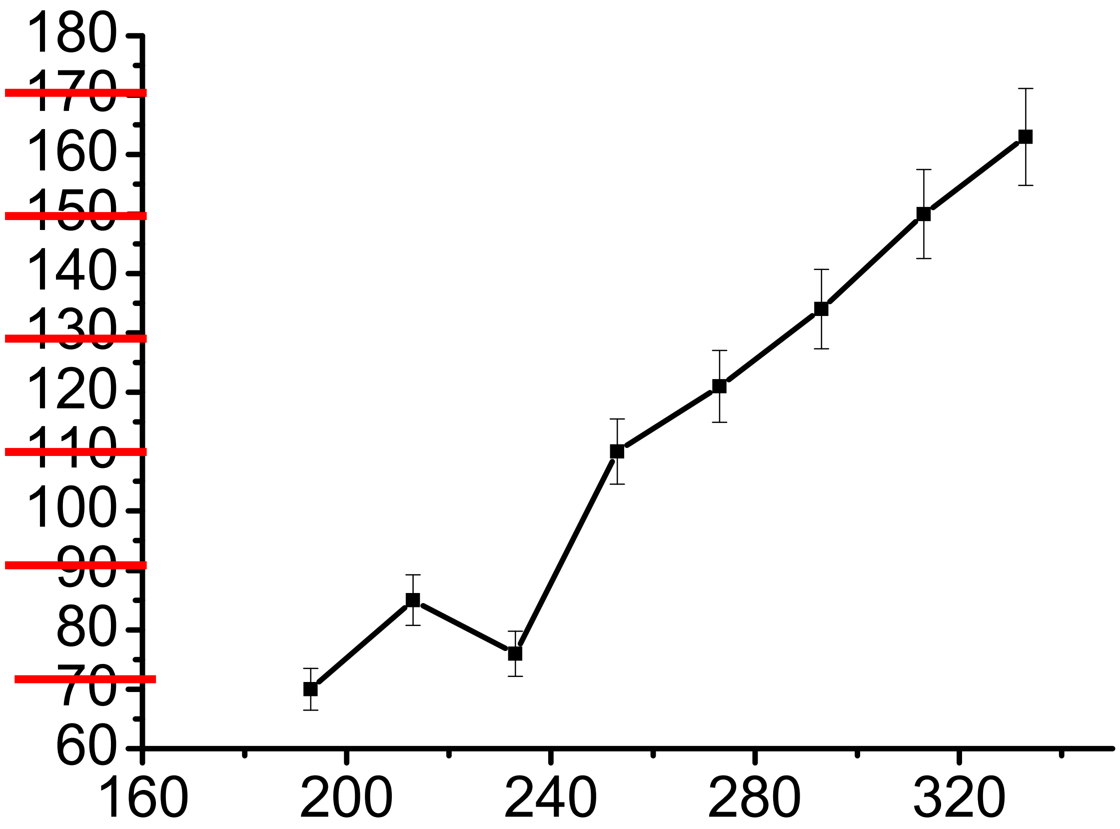
# 2. Measurement uncertainty is missing

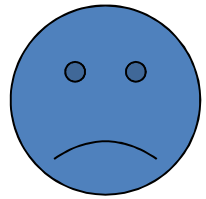


# 3. Adjust the axis to the range of experimental data!!!



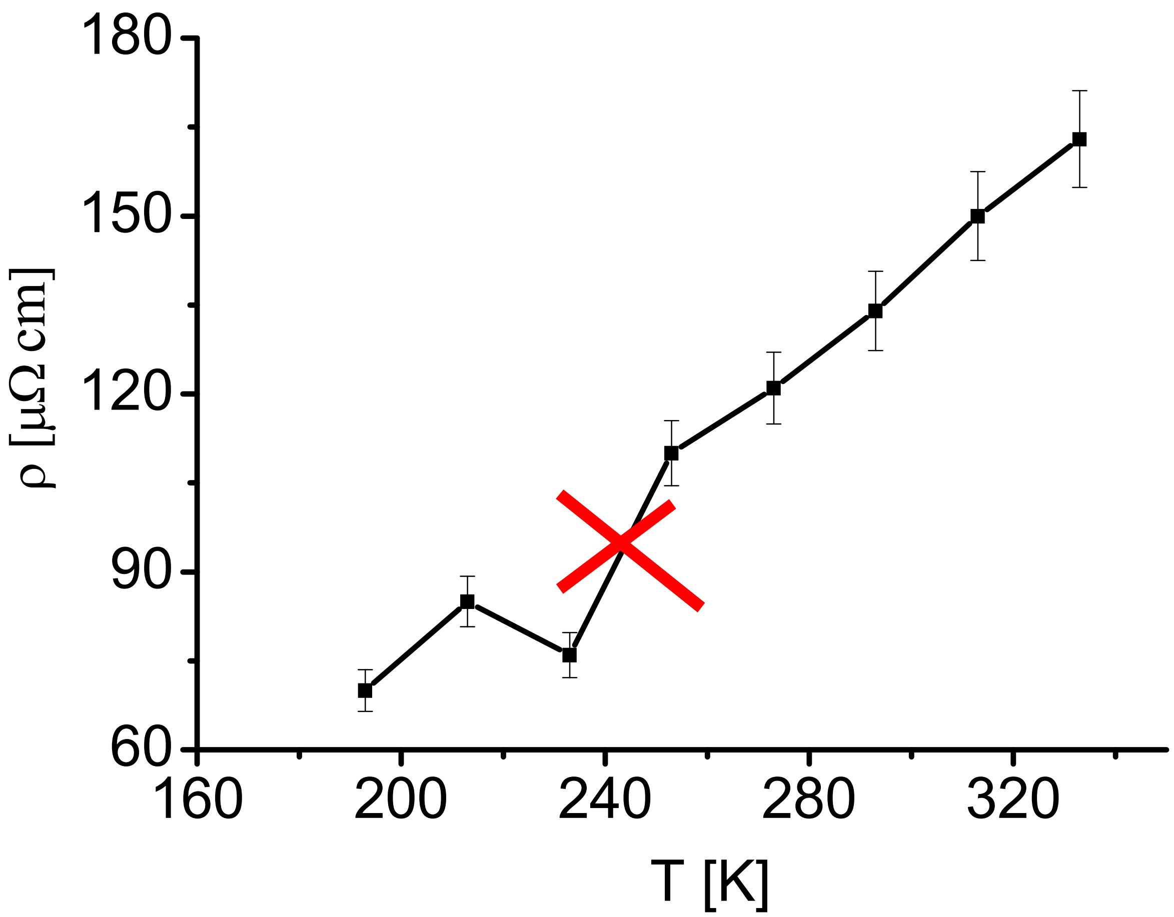
# 4. Properly describe the axes and choose the scale in order to read the data easily.

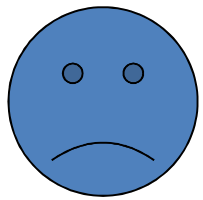


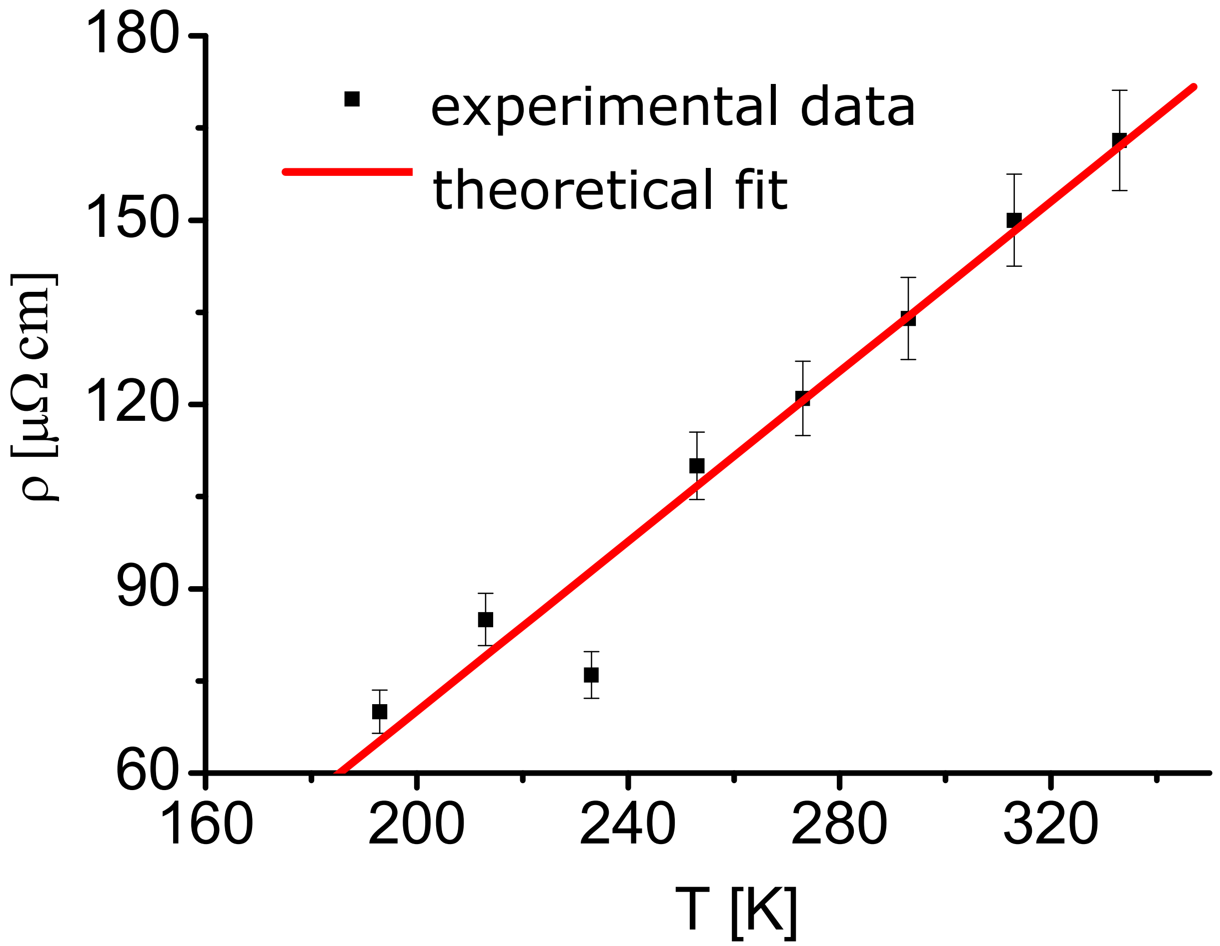
***What quantity is represented by this axis???***

**5. Do not connect the experimental points by polygonal chains!!! If the theoretical model is known, it is advised to make a fit to the experimental data.**



# 6. Take care of the esthetic aspect of your plot (legend, frame, etc.)



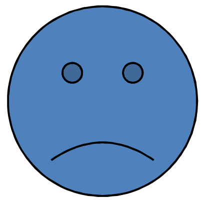
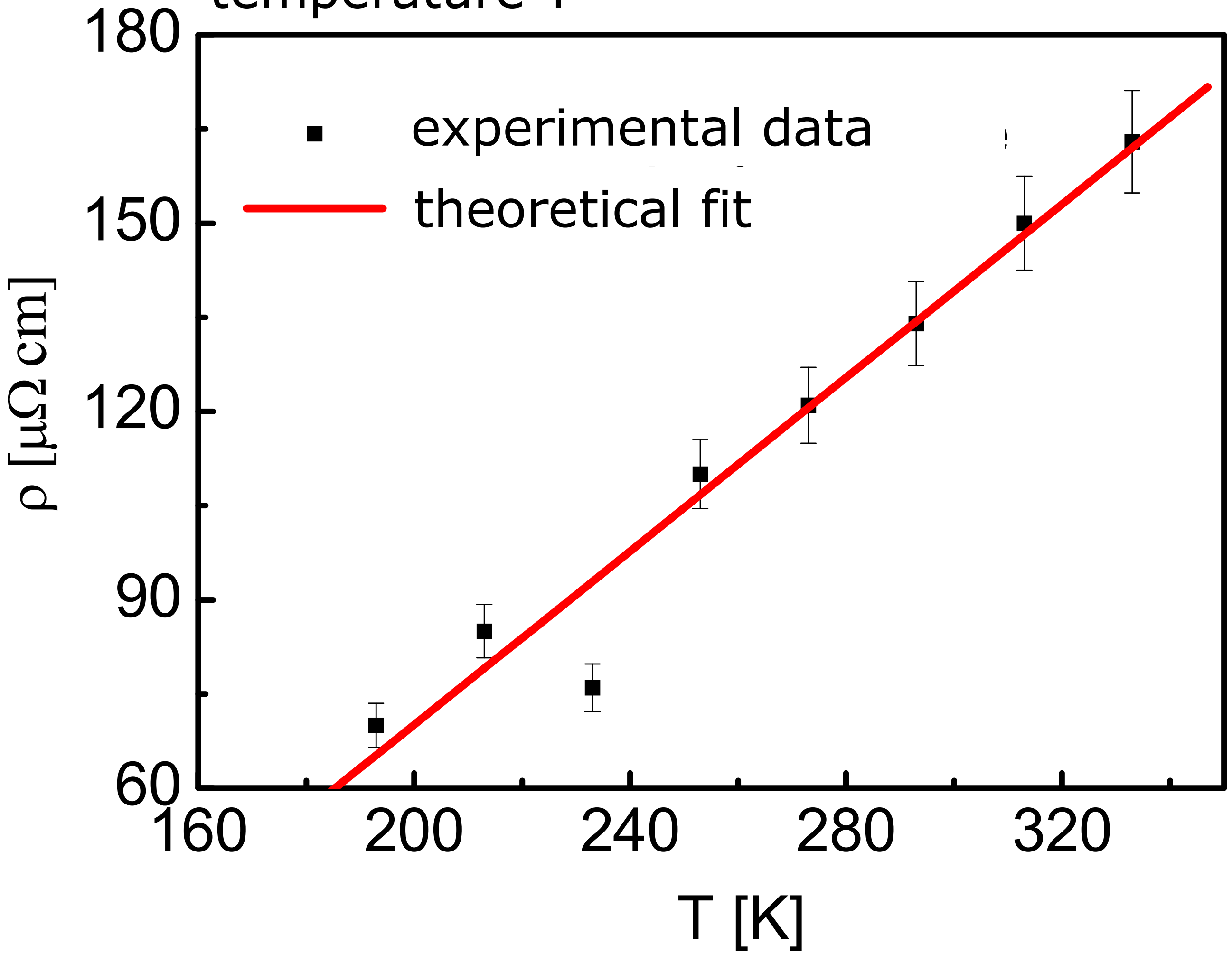
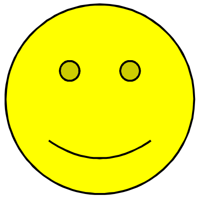



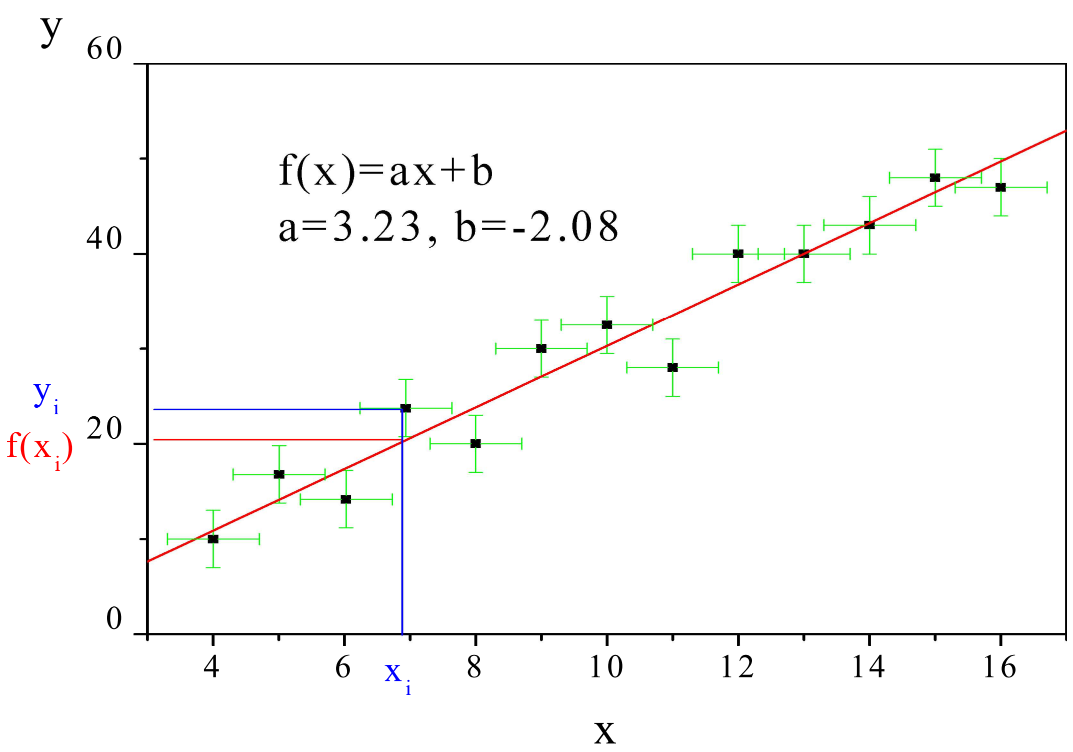
Fig.1

Resistivity ρ of Bi sample as a function of temperature T



# Least Square Method - Linear Regression




$$S^2 = \sum_i^n \left[y_i - (ax_i + b)\right]^2 = \min$$

# Minimum of two-variable function:

$$\frac{\partial S^2}{\partial a} = 0 \qquad \frac{\partial S^2}{\partial b} = 0$$

A set of two linear equations is obtained for unknown *a* and *b*

$$a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$a\sum x_i + bn = \sum y_i$$

Solution gives us the following expressions for *a* and *b*

$$a = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{W}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{W}$$

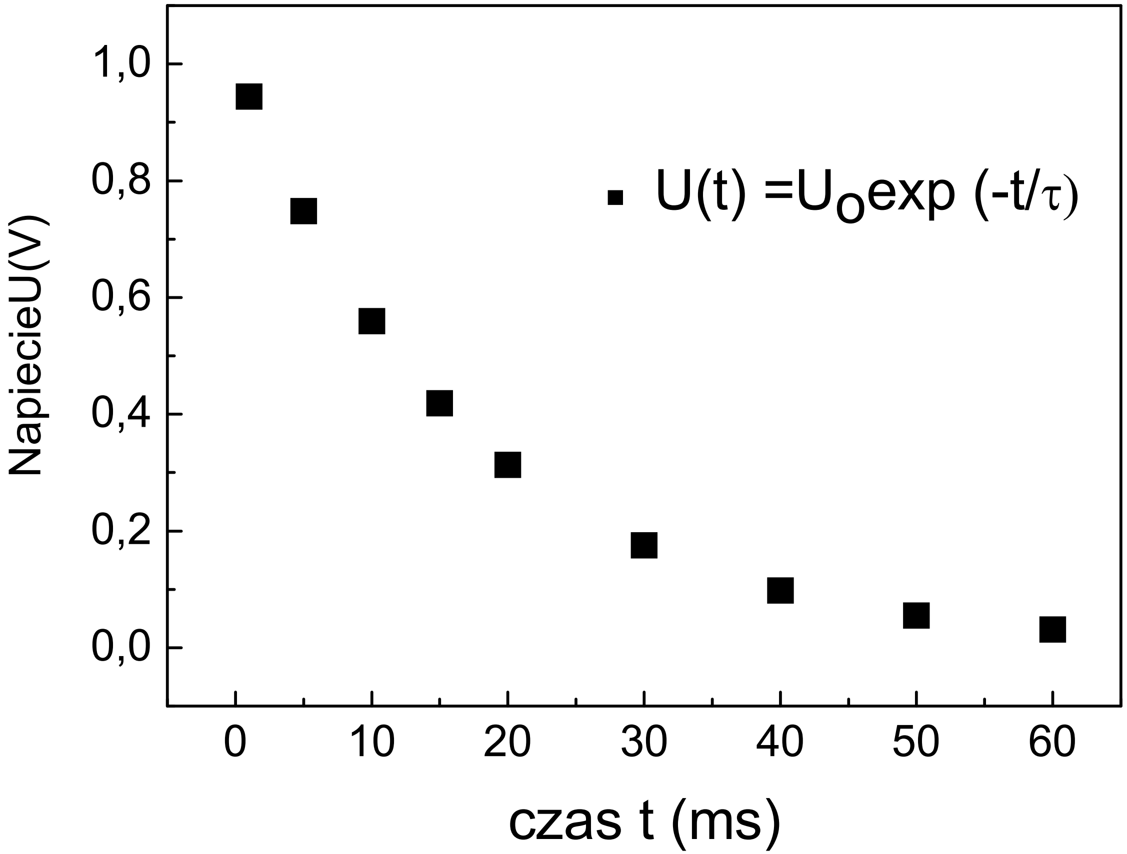
$$W = n\sum x_i^2 - \left(\sum x_i\right)^2$$

***Standard deviations of a and b can be determined from the laws of statistics:***

$$u(a) = \sqrt{\frac{n}{n-2}}\sqrt{\frac{S^2}{W}}$$

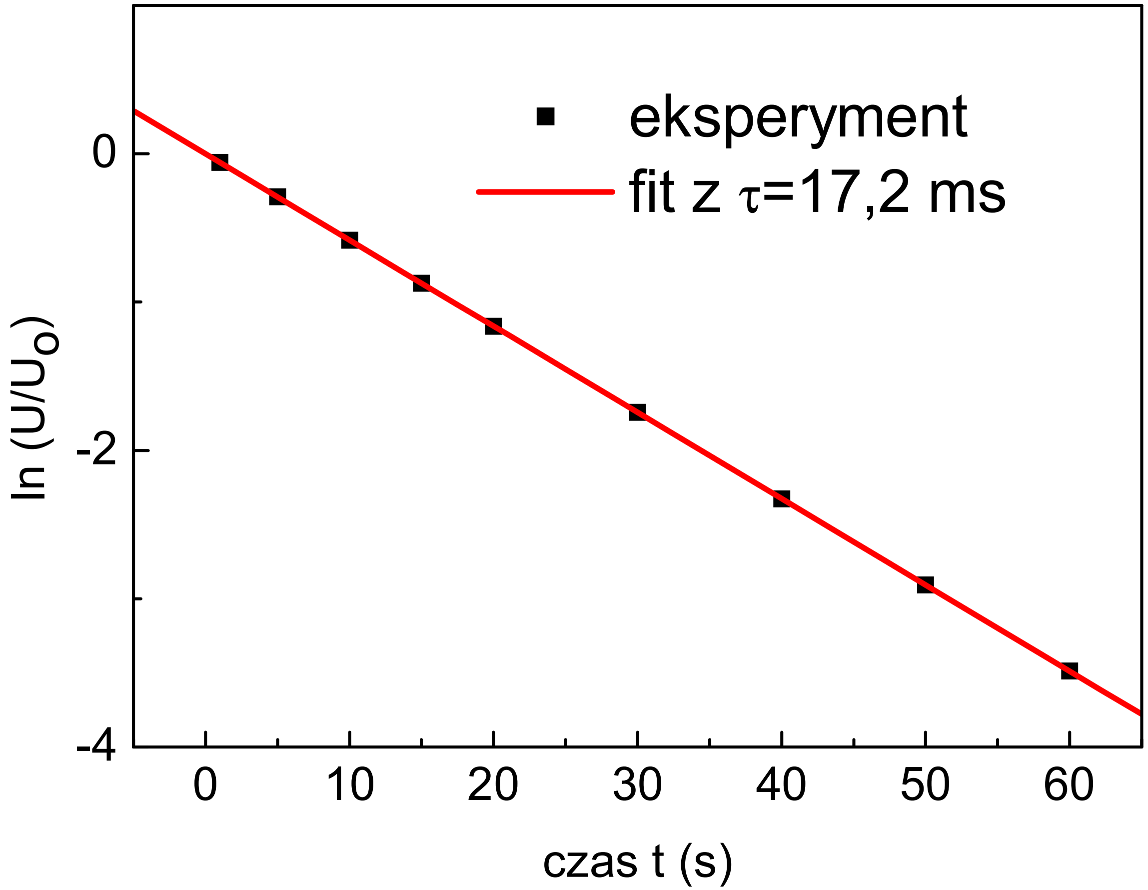
$$u(b) = u(a)\sqrt{\frac{\sum x_i^2}{n}}$$

# Linearization of data



Line can be fitted after tranformation of data to

$\ln(U/U_o) = -t/\tau$



# USEFUL HINTS

1. **Results of laboratory measurements** suffer from **uncertainties**, that the researcher is obliged to **estimate** according to certain **rules**.

2. In the first place, one has to find all possible **sources of errors**, keeping in mind that results with gross errors should not be taken into account. **In student laboratory systematic errors usually mask random errors.**

3. **Multiple repetitions of measurement do not make sense when the systematic error predominates.** In this case one should perform up to 3-5 measurements under the same conditions in order to make sure that the results are reproducible.

# USEFUL HINTS

4. When **random events** are the main source of **errors**, it is necessary to make sure that distribution of results can be described by Gauss function. If not, should one expect some other distribution function? In order to solve this problem one has to repeat the measurements (e.g. 100 times) under the same conditions, calculate the average and variance, draw a histogram, etc.

5. As a measure of uncertainty use rather **standard uncertainty**, scarcely maximum uncertainty.

6. In the case of **complex measurand**, one should apply **laws of error propagation**. An effort should be made in order to estimate the **contributions** to the total value of error coming from measurements of simple measurands. In order to achieve this goal one has to calculate **relative uncertainties**.

# USEFUL HINTS

7. **Graph** is quite important part of **lab report** (not only in the student's laboratory). Graphs should be prepared according to **certain rules**, unambiguous description is required.

8. If a **theoretical model** of phenomenon under study is known, one should place a theoretical curve (continuous line) upon clearly distinguished experimental points (right size symbols should be chosen; experimental cross-bar errors should be included). Well-known methods of fitting should be applied.

9. Whenever possible, we can perform linearization of data, plotting e.g., y vs. ln (x), or log y vs. log x, or y vs. 1/x etc. To data prepared in such a way one can apply a method of linear regression.